Units

You often solve problems in science by doing arithmetic on *measurements*. A measurement is not just a number; it is more than a number. A measurement has two parts: a number and a name, called the measurement's *unit*.

Correct arithmetic on measurements *requires* that you include the units in the arithmetic. You do this by treating the unit name just like a numeric *factor*.

Measurements

The data of science consist of measurements. A measurement is written or spoken in two parts: a number and a name; the name part is called the measurement's unit (sometimes the word *dimension* is used instead of unit). Examples of measurements are

3 feet, 0.5 liter, 2200 grams.

"Feet", "liters", and "grams" are units.

In a measurement *the number part without the unit part is meaningless*. You can see this by noticing that both "1 mile" and "5280 ft" are the same measurement. Since both "1 mile" and "5280 ft" describe the same measurement you can write the equation

$$1 \text{ mile} = 5280 \text{ ft}$$

but of course 1 is not equal to 5280. If you don't treat the unit as something equally important as the number there is a strong chance that you will be led astray in your computation.

Measurements Occurring in the Study of Mechanics

Three basic things are measured in mechanics: length, mass, and time. There is an international standard set of units for these types of measurements; it is called **SI** (for Système Internationale), and we shall use it. The United States has been slow to adopt SI for popular use, although businesses that trade outside the US must use it in their own self-interest. This table shows the names and abbreviations of the three basic types of measurements in both SI and US systems.

Measurement Type	SI unit	SI abbrev.	US unit	US abbrev.
Length	meter	m	foot	ft
Mass	kilogram	kg	pound	lb
Time	second	s	second	s

The Algebra of Measurements

Factors are numbers that multiply together. The factors of a particular number are numbers that can multiply together to make this number. For example, 2, 3, 4, and 6 are factors of 12.

A unit word is a factor of the measurement in which it appears. For example, the factors of the measurement

12 in (12 inches, or one foot)

are

2, 3, 4, 6, and "in".

In "12 in" you are multiplying the number "12" and the unit "in". If this seems weird it's because a measurement is more than just a number, and *it's not correct to think of a measurement as just a number*. (Just ignore whether a unit word is written singular or plural; from now on we'll favor abbreviations for consistency.)

Thinking of a unit as a factor is all we need in order to understand the algebra of measurements. Here are the rules.

Addition Rule. In order to add or subtract two measurements, their units must be the same, i.e., "you can't add apples and oranges".[1]

Multiplication Rule. When multiplying and dividing measurements, the units must be treated as factors in the computation. That is, units can be multiplied and canceled. You will see examples below.

Application of the Multiplication Rule: Area and Volume

Length, area, and volume are measurements that illustrate the multiplication rule. Here is an example. The length and width of a rectangle are length measurements. The rule for computing the area of a rectangle is length · width, that is, the product of two lengths. Say the length measurement is 6 cm and the width measurement is 5 cm. (Note: 1 m = 100 cm.) Here is the area computation.

$$\text{area} = \text{length} \cdot \text{width} = 6\text{ cm} \cdot 5\text{ cm} = 30\text{ cm} \cdot \text{cm} = 30\text{ cm}^2$$

Here you see that "cm^2" is not just shorthand for saying "square centimeters"; it is actually the result of treating units as factors and multiplying cm by cm. *This is not a coincidence; it is how these things work.* That is, "cm^2" is a unit of area. *A unit of area is a product of two length units.* Examples are cm^2 and in^2; even cm · mile would be a valid area unit.

The same idea holds for volume units. *A unit of volume is a product of three length units.* Examples are m^3, cm^3, and in^3. (Here is a curious real-world example: acre · ft is used to measure water volume in agricultural irrigation.)

Application of the Multiplication Rule: Unit Conversion

Unit conversion is the process of changing a measurement with one unit into a measurement with another unit. It is a more powerful technique than is generally appreciated, for two reasons.

- It can be used in a wider class of problems than conversion of units in measurements. For example, it is a key to solving rate problems.
- You can use the technique to *guide* your setup of many problems to the point that the setup process requires little thought.

[1] Here is the proof. Use the distributive property of numbers.

$$5\text{ cm} + 6\text{ cm} = (5 + 6)\text{ cm} = (11)\text{ cm} = 11\text{ cm}.$$

You can't do this if you mix cm and ft, for example.

We start with an equation that declares the equality of two measurements, for example,

$$1\,\text{mole} = 6.022 \bullet 10^{23}\ \text{particles}$$

(We use “mole” as shorthand for “mole of particles”.) Using the rule from algebra that we can perform the same operation (a division) on both sides of an equation we arrive at the following two identities.

$$1 = \frac{6.022 \bullet 10^{23}\ \text{particles}}{1\,\text{mole}}$$

and

$$1 = \frac{1\,\text{mole}}{6.022 \bullet 10^{23}\ \text{particles}}$$

These two fractions both have the value 1. (Yet, they are reciprocals; 1 is the only number that is its own reciprocal.) Here is the key to this method: *You can multiply any measurement by 1 and the product will be the same measurement*.

Since the words “mole” and “particle” are factors of the measurements in which they occur, we are free to multiply and cancel them.

In this example we’ll use “Cheerios” instead of “particles”. How many Cheerios are in 3.6 mole of Cheerios? We start with what we are given,

$$3.6\ \text{mole}$$

and multiply by 1, using the form of 1 that we want:

$$1 = \frac{6.022 \bullet 10^{23}\ \text{Cheerios}}{1\ \text{mole}}$$

This gives:

$$3.6\ \text{mole} = 3.6\ \text{mole} \bullet 1 = 3.6\ \text{mole} \bullet \frac{6.022 \bullet 10^{23}\ \text{Cheerios}}{1\ \text{mole}} = 3.6 \bullet 6.022 \bullet 10^{23}\ \text{Cheerios}$$

Let us now be more systematic about it. If the question is: “How many Cheerios are in 3.6 mole of Cheerios?” that means we are starting with 3.6 mole of Cheerios and we want to change it to some other form. So the first step is always to start with the identity

$$3.6\ \text{mole} = 3.6\ \text{mole}\ .$$

Then we leave the left side alone and we multiply the right side by 1, as many times as we want and using as many different forms of 1 as we choose, with the sole purpose being to *cancel out the words we don’t want and introduce the words we do want*.

Thus (follow the numbered steps):

1: $$3.6 \text{ mole} = 3.6 \text{ mole}$$

2: $$3.6 \text{ mole} = 3.6 \text{ mole} \cdot 1$$

now we substitute the form of 1 that comes from

3: $$1 \text{ mole} = 6.022 \bullet 10^{23} \text{ Cheerios}$$

so we can cancel the "mole" unit, producing

4: $$3.6 \text{ mole} = 3.6 \text{ mole} \cdot \frac{6.022 \bullet 10^{23} \text{ Cheerios}}{1 \text{ mole}}$$

then we cancel the "mole" units

5: $$3.6 \text{ mole} = 3.6 \cancel{\text{mole}} \cdot \frac{6.022 \bullet 10^{23} \text{ Cheerios}}{1 \cancel{\text{mole}}}$$

and we get

6: $$3.6 \text{ mole} = 3.6 \cdot 6.022 \bullet 10^{23} \text{ Cheerios}$$

and finally

7: $$3.6 \text{ mole} = 21.68 \bullet 10^{23} \text{ Cheerios}$$

or, following the usual rules for "normalizing" scientific notation,

8: $$3.6 \text{ mole} = 2.168 \bullet 10^{24} \text{ Cheerios}$$